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Nonuniqueness of the Phase Shift in Central Scattering due to Monodromy

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Dullin and Waalkens Reply: The results of our Letter [1] are correct. In the Letter we showed that the phase shift for 2D central scattering at a smooth repulsive potential cannot be uniquely defined as a globally *smooth* function of angular momentum and energy due to a topological obstruction similar to monodromy in bound systems.

The preceding Comment by Eiglsperger *et al.* [2] raises two points. First of all, the authors of the Comment point out that the phase shift can be uniquely defined as a *nonsmooth* function. Second, the authors claim that the WKB approximation is bad for the class of potentials studied in our Letter. The first point is trivial and no contradiction to what we say in our Letter. We suppose that the commentators raise this point because we did not mention the word "smooth" in the title and abstract of our Letter. We stress, however, that the entire discussion in our Letter is based on and is only meaningful for globally *smooth phase shifts*. The smoothness assumption is explicitly stated in the paragraph before Eq. (3) in our Letter, and also implied by the term *monodromy* in the title. In the second point the authors give an example of a potential for which the WKB approximation for s-wave scattering is bad near the critical threshold energy $E = 0$. It is well known the WKB approximations become bad near critical values. However, it is the essence of monodromy that, for the smooth continuation of the phase shift, one can stay away from critical values. Accordingly, we will show below that for the relevant noncritical energies, the WKB approximation works very well also for the "counterexample" in the Comment [2].

First point.—The key point of our Letter is that globally ΔW cannot be defined as a single-valued *smooth* function of l *and* $p = (2mE)^{1/2}$. This is surprising, because *locally* ΔW *can* be defined as such a single-valued smooth function. Two-dimensional central scattering is integrable. The angular momentum $L = xp_y - yp_x$ and the Hamiltonian $H = \frac{1}{2m}(p_x^2 + p_y^2) + V(x^2 + y^2)$ are constants of motion in involution with linearly independent gradients almost everywhere in the phase space $\mathbb{R}^4$. In fact, for a smooth repulsive central potential of the type discussed in our Letter, the only points in the phase space at which ∇L and ∇H are linearly dependent are the origin, with corresponding critical value $(L, H) = (0, E_c)$ where $E_c = V(0)$, and the trivial equilibrium points at infinity where the particle is at rest with critical value $E = 0$. In a neighborhood of a regular value (l, E) the Liouville-Arnold theorem guarantees that ΔW is a smooth function of l and E. However, these *local* definitions of ΔW cannot be patched together to give a *global* single-valued smooth function ΔW, i.e., a function which is smooth and single-valued on the set of regular values of L and H. The function ΔW near $(L, H) = (0, E_c)$ has properties that are similar to the function $\Re(z \log z - z)$ with $z = (H - E_c) + iL \in \mathbb{C}$ near zero. Analytic continuation leads to a smooth but multivalued function. Of course, if one gives up the smoothness of ΔW and introduces a branch cut as done in the Comment [2], then ΔW becomes unique. This trivial statement, however, has no bearing whatsoever on the result of our Letter.

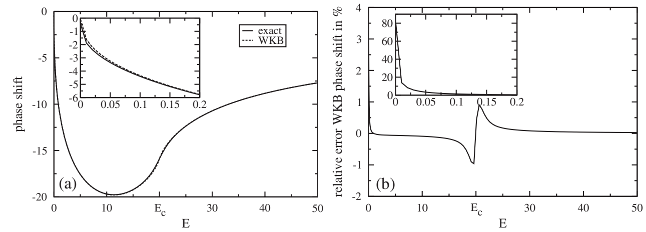

FIG. 1. (a) Exact phase shift and the corresponding WKB approximation for the potential $V(r) = 20\exp(-r^2)$. ($m = 1$, $\hbar = 0.25$.) (b) Error of the WKB phase shift.

Second point.—In Fig. 1 we compare the (exact) phase shift δ with its WKB approximation ΔW for s-waves scattering ($l = 0$) at the potential $V(r) = 20\exp(-r^2)$ considered in the Comment [2]. For energies very close to the critical threshold $E = 0$ the WKB approximation is bad as pointed out in the Comment [2]. This behavior is related to the fast decay of the potential at infinity. In the regular energy range relevant for the analytic continuation around the critical value $(L, H) = (0, E_c) = (0, 20)$, however, the error of the WKB approximation is far below the stated 1% in our Letter.

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[1] H. R. Dullin and H. Waalkens, Phys. Rev. Lett. **101**, 070405 (2008).
[2] J. Eiglsperger, H. Friedrich, and J. Madroñero, preceding Comment, Phys. Rev. Lett. **102**, 188901 (2009).

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